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A Fiber Distributed Model of Biological Tissues

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Abstract

Several micro-structured biological tissues are characterized by anisotropy. The dependence of stiffness and strength of the material on the direction is determined mainly by the presence of cable-like micro- and nano-structures made of collagen. Recent findings concerning the arrangement of the structural collagen in biological tissues suggest that, although the functionality would require a prevailing orientation of the fibers, the organization of the organ introduces instead a certain degree of dispersion. In this regard, we propose a material model alternative to the one based on generalized structure tensors, proposed by Gasser et al. (2006). In the present model the strain energy function is assumed to be dependent on the mean value and on the variance of the pseudo-invariant I_4 of the fiber distribution. We consider the stress response under standard uniaxial shear and biaxial loading conditions of the proposed model. Finally, we derive an approximated explicit expression of the covariance tensor for the second Piola-Kirchhoff stress and verify such expression via numerical integration.

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Keywords: biological tissue; collagen fibrils; Von Mises distribution; second Piola-Kirchoff covariance stress tensor

1. Introduction

The modeling of biological tissues often requires to consider the presence of a hierarchical organization of collagen proteins embedded into a matrix with isotropic properties. Nano-structured cable-like collagen proteins define the basic unit that specialize in micro-fibrils. Fibrils in turn organize as one-dimensional cables, or, by aggregation of two or more fibril sets, as two-dimensional sheets. The overall arrangement of the fibrils, or more in general, of the fibers, is such that the resulting biological structure assumes a one-dimensional (e. g., tendons) or a two-dimensional shape (e. g., artery wall, cornea or sclera shell, and others). While in one-dimensional structures the geometry requires the full alignment of the fibers in one direction, in two-dimensional structures the organization of the fibril distribution is not certain, and a certain degree of dispersion in the orientation of the fibers is in general observed, in particular when the structure is thin and shell like and a containment function is necessary in all directions. Furthermore, in thicker structures the dispersion of the fiber orientation may assume a fully three-dimensional character. In recent times,

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the need to define accurate predictive models of the mechanical behavior of biological tissues gave rise to numerous proposal of fiber distributed anisotropic models, see, e. g., [1, 10, 2, 3, 5, 7].

2. Material Model

In the framework of nonlinear continuum mechanics, we postulate the existence of a Helmholtz free-energy function Ψ defined per unit reference volume. We comply with the case in which the free energy is assumed to be dependent on the deformation gradient $\mathbf{F}$ only, i. e., $\Psi = \Psi(\mathbf{F})$. For a biological tissue with collagen fibers it is customary to decompose additively the strain energy into three terms:

$$\Psi = \Psi_{\text{vol}} + \Psi_{\text{iso}} + \Psi_{\text{aniso}}. \tag{1}$$

The first term, Ψ_{vol}, accounts for volume changes, and it is assumed to be dependent on the volumetric part of the deformations, i. e., on the jacobian $J = \det\mathbf{F}$, i. e.,

$$\Psi_{\text{vol}} = \Psi_{\text{vol}}(J). \tag{2}$$

The second term, Ψ_{iso}, accounts for the isotropic behavior of the material due to the underlying matrix and to a portion of randomly distributed fibrous reinforcement. Usually, Ψ_{iso} is assumed to be dependent on the first and second invariants, $\bar{I}_1$ and $\bar{I}_2$, of the modified Cauchy-Green deformation tensor $\overline{\mathbf{C}} = \overline{\mathbf{F}}^T\overline{\mathbf{F}}$, where $\overline{\mathbf{F}} = J^{-1/3}\mathbf{F}$:

$$\Psi_{\text{iso}} = \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2). \tag{3}$$

The effect of the fibrous reinforcement is described by the third term Ψ_{aniso}. Often, it is assumed to be dependent on the modified tensor $\overline{\mathbf{C}}$ and on particular vectors –or tensors– describing the intrinsic microstructure of the material. As a consequence of the additive decomposition of (1) and of the decoupling of the arguments between the terms, it follows that the second Piola-Kirchhoff stress tensor $\mathbf{S}$ is in turn the sum of three terms:

$$\mathbf{S} = \mathbf{S}_{\text{vol}} + \mathbf{S}_{\text{iso}} + \mathbf{S}_{\text{aniso}}, \tag{4}$$

in the form:

$$\mathbf{S} = 2\frac{\partial \Psi}{\partial \mathbf{C}} = 2\frac{\partial \Psi_{\text{vol}}}{\partial \mathbf{C}} + \left(\overline{\mathbf{S}}_{\text{iso}} + \overline{\mathbf{S}}_{\text{aniso}}\right)\frac{\partial \overline{\mathbf{C}}}{\partial \mathbf{C}}, \tag{5}$$

where:

$$\overline{\mathbf{S}}_{\text{iso}} = 2\frac{\partial \Psi_{\text{iso}}}{\partial \overline{\mathbf{C}}}, \qquad \overline{\mathbf{S}}_{\text{aniso}} = 2\frac{\partial \Psi_{\text{aniso}}}{\partial \overline{\mathbf{C}}}. \tag{6}$$

Also the fourth order elasticity tensor derives as the sum of three terms:

$$\mathbb{C} = 4\frac{\partial^2 \Psi}{\partial \mathbf{C} \partial \mathbf{C}} = \mathbb{C}_{\text{vol}} + \mathbb{C}_{\text{iso}} + \mathbb{C}_{\text{aniso}}. \tag{7}$$

where the anisotropic elasticity tensor is defined as

$$\mathbb{C}_{\text{aniso}} = 4\frac{\partial^2 \Psi_{\text{aniso}}}{\partial \mathbf{C} \partial \mathbf{C}}, \tag{8}$$

where one of the components is (cf. [9]):

$$\overline{\mathbb{C}}_{\text{aniso}} = 4J^{-4/3}\frac{\partial^2 \Psi_{\text{aniso}}}{\partial \overline{\mathbf{C}} \partial \overline{\mathbf{C}}}. \tag{9}$$

Explicit formulae for the second Piola-Kirchhoff stress and for the elasticity tensors can be found in [9].
According to [6, 8], in the case of a single family of parallel fibers oriented in the referential direction $\mathbf{a}$, a convenient form of the anisotropic Helmholtz free energy is given by

$$\Psi_{\text{aniso}}(\bar{I}_4) = \overline{\Psi}_{\text{aniso}}(\bar{I}_4) + \Psi^0_{\text{aniso}} = \frac{k_1}{2k_2}\exp\left[k_2\left(\bar{I}_4 - 1\right)^2\right] - \frac{k_1}{2k_2}, \tag{10}$$

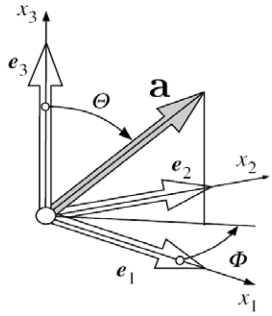

Fig. 1. Definition of a unit vector in a spherical reference system.

where the pseudo-invariant $\overline{I}_4$ is the contraction of the modified Cauchy-Green deformation tensor $\overline{\mathbf{C}} = \overline{\mathbf{F}}^T\overline{\mathbf{F}}$ and of the second order tensor $\mathbf{A} = \mathbf{a} \otimes \mathbf{a}$, i. e.,

$$\overline{I}_4(\mathbf{A}, \overline{\mathbf{C}}) = \mathbf{A} : \overline{\mathbf{C}}. \tag{11}$$

Since the contribution of the fibers is accounted for only in extension, the anisotropic contribution to stress and elasticity tensors must be considered if and only if $\overline{I}_4 > 1$, otherwise it must be set to zero.

3. Distributed Model

Within the unit sphere ω centered at a material point, the unit vector $\mathbf{a}$ can be expressed in terms of two spherical angles:

$$\mathbf{a}(\Theta, \Phi) = \sin\Theta\cos\Phi\,\mathbf{e}_1 + \sin\Theta\sin\Phi\,\mathbf{e}_2 + \cos\Theta\,\mathbf{e}_3, \tag{12}$$

where $\Theta \in [0, \pi]$ and $\Phi \in [0, 2\pi]$, see Fig. 1. Thus, the structure second order tensor $\mathbf{A} = \mathbf{a} \otimes \mathbf{a}$ becomes:

$$\mathbf{A} = \begin{bmatrix} \sin^2\Theta\cos^2\Phi & \sin^2\Theta\sin\Phi\cos\Phi & \sin\Theta\cos\Theta\cos\Phi \\ \sin^2\Theta\sin\Phi\cos\Phi & \sin^2\Theta\sin^2\Phi & \sin\Theta\cos\Theta\sin\Phi \\ \sin\Theta\cos\Theta\cos\Phi & \sin\Theta\cos\Theta\sin\Phi & \cos^2\Theta \end{bmatrix}. \tag{13}$$

Let us assume that the orientation of the fibers is spatially distributed according to a density function $\rho(\mathbf{a})$, satisfying the obvious symmetry requirement $\rho(\mathbf{a}) \equiv \rho(-\mathbf{a})$. Therefore, the amount $\rho(\mathbf{a})\sin\Theta d\Theta d\Phi$ defines the number of fibers whose orientation falls in the range $[(\Theta, \Theta + d\Theta), (\Phi, \Phi + d\Phi)]$. For the unit sphere ω the following property holds:

$$\int_\omega \rho(\mathbf{a})d\omega = \int_0^\pi \int_0^{2\pi} \rho(\mathbf{a})\sin\Theta d\Phi d\Theta = 4\pi. \tag{14}$$

In a simplified but rather realistic case, the fibers are distributed with rotational symmetry about a mean referential direction $\mathbf{a}_0$. Therefore the fibers confer to the material a *transversally isotropic character*. With no loss of generality, direction $\mathbf{a}_0$ can be taken parallel to the basis vector $\mathbf{e}_3$, so that the density can be taken independent of Φ, thus $\rho(\mathbf{a}) \equiv \rho(\Theta)$, and the normalization condition (14) reduces to

$$\int_0^\pi \rho(\Theta)\sin\Theta d\Theta = 2. \tag{15}$$

For an assigned density ρ, it is possible to introduce the average operator $< \cdot >$, related to the considered distributed orientation of the fibers, as

$$< \cdot > = \frac{1}{4\pi}\int_\omega \rho(\mathbf{a})(\cdot)\,d\omega. \tag{16}$$

In particular, the average pseudo invariant $\overline{I}_4$ becomes:

$$\overline{I}_4^* = <\overline{I}_4> = <\mathbf{A} : \overline{\mathbf{C}}> = \frac{1}{4\pi}\int_\omega \rho(\mathbf{a})(\mathbf{A} : \overline{\mathbf{C}})\, d\omega = <\mathbf{A}> : \overline{\mathbf{C}}. \tag{17}$$

The tensor $<\mathbf{A}>$ that appears in the expression of the average $\overline{I}_4$ coincides with the symmetric *generalized structure second order tensor* (GST) $\mathbf{H}$ introduced in [4]:

$$\mathbf{H} = <\mathbf{A}> = \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1-2\kappa \end{bmatrix}, \tag{18}$$

where the constant κ is

$$\kappa = \frac{1}{4}\int_0^\pi \rho(\Theta)\sin^3\Theta d\Theta. \tag{19}$$

The associated average strain energy density is

$$\Psi_{\text{aniso}}^* = <\Psi_{\text{aniso}}> = \frac{1}{4\pi}\int_\omega \rho(\mathbf{a})\Psi_{\text{aniso}}(\overline{I}_4)\, d\omega. \tag{20}$$

For general choices of the density ρ, the evaluation of (20) requires the use of numerical quadrature formulas; therefore it is not possible to derive the explicit expressions of the stress and elasticity tensors, but they must be computed through numerical quadrature as well.

An interesting interpretation of the anisotropic strain energy discussed in [4] is provided by writing the first order term of the Taylor series of (10) expanded around the mean argument $\overline{I}_4^*$:

$$\Psi_{\text{aniso}}^* \approx \Psi_{\text{aniso}}(\overline{I}_4^*) + \left.\frac{\partial \Psi_{\text{aniso}}}{\partial \overline{I}_4}\right|_{\overline{I}_4=\overline{I}_4^*} <\overline{I}_4 - \overline{I}_4^*> = \overline{\Psi}_{\text{aniso}}(\overline{I}_4^*) + \Psi_{\text{aniso}}^0 = \overline{\Psi}_{\text{aniso}}^* + \Psi_{\text{aniso}}^0. \tag{21}$$

The linear term in (21) because $<\overline{I}_4 - \overline{I}_4^*>$, see (17). In [9] we proposed a better approximation of the average anisotropic strain energy by accounting for the second order term of the Taylor's expansion as

$$\Psi_{\text{aniso}}^* \approx \Psi_{\text{aniso}}(\overline{I}_4^*) + \frac{1}{2}\left.\frac{\partial^2 \Psi_{\text{aniso}}}{\partial \overline{I}_4{}^2}\right|_{\overline{I}_4=\overline{I}_4^*} <(\overline{I}_4 - \overline{I}_4^*)^2> = \Psi_{\text{aniso}}^0 + \overline{\Psi}_{\text{aniso}}^*\left(1 + K^*\sigma_{\overline{I}_4}^2\right) \tag{22}$$

where we denote

$$K^* = K(\overline{I}_4) = k_2\left[2k_2\left(\overline{I}_4 - 1\right)^2 + 1\right] \tag{23}$$

and

$$\sigma_{\overline{I}_4}^2 = \overline{\mathbf{C}} : \mathbb{H} : \overline{\mathbf{C}} - (\mathbf{H} : \overline{\mathbf{C}})^2. \tag{24}$$

The non zero elements of fourth order tensor $\mathbb{H} = <\mathbf{A} \otimes \mathbf{A}>$ are

$$\begin{array}{rcl} H_{1111} = H_{2222} & = & 3\widehat{\kappa}, \\ H_{3333} & = & 8\widehat{\kappa} + 1 - 4\kappa, \\ H_{1122} = H_{2211} = H_{1212} = H_{2121} = H_{1221} = H_{2112} & = & \widehat{\kappa}, \\ H_{2233} = H_{3322} = H_{2323} = H_{3232} = H_{2332} = H_{3223} & = & -4\widehat{\kappa} + \kappa, \\ H_{3311} = H_{1133} = H_{3131} = H_{1313} = H_{3113} = H_{1331} & = & -4\widehat{\kappa} + \kappa, \end{array} \tag{25}$$

and $\widehat{\kappa}$ is

$$\widehat{\kappa} = \frac{1}{16}\int_0^\pi \rho(\Theta)\sin^5\Theta d\Theta. \tag{26}$$

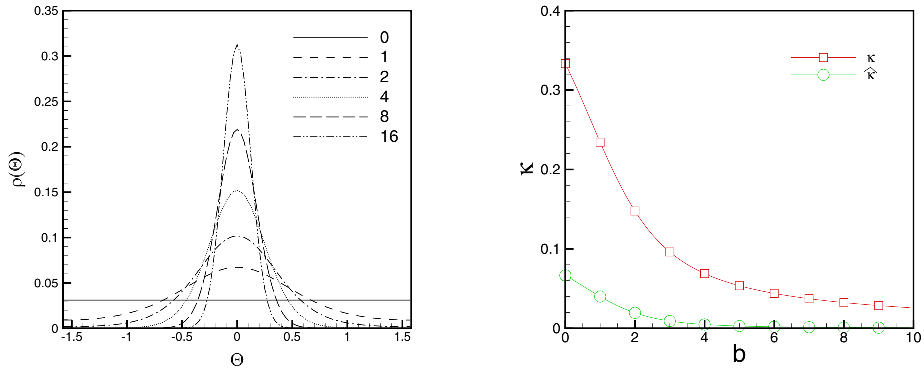

Fig. 2. (a) Meaning of the parameter b in the normalized von Mises distribution. Each curve corresponds to a different value of b, as described in the legend. (b) Dependence of the parameters κ and $\widehat{\kappa}$ from the parameter b in the normalized von Mises distribution.

Note that the previous formulae refer to a particular orientation of the mean direction of the fibers, assumed here to be coincident with the unit vector $\mathbf{e}_3$. In applying the model to a generic orientation $\mathbf{a}_0$, the fourth order tensor $\mathbb{H}$ has to be rotated. To this purpose, let θ be the angle between the two unit vectors $\mathbf{a}_0$ and $\mathbf{e}_3$ such as

$$\cos\theta = \mathbf{e}_3 \times \mathbf{a}_0. \tag{27}$$

The two vectors define a plane of normal $\mathbf{n}$, i.e.

$$\mathbf{n} = \frac{\mathbf{e}_3 \times \mathbf{a}_0}{|\mathbf{e}_3 \times \mathbf{a}_0|}. \tag{28}$$

A rotation of amplitude θ about the vector $\mathbf{n}$, of components n_i, is governed by the matrix $\mathbf{R} \in SO(3)$, readily:

$$\mathbf{R} = \begin{bmatrix} c + n_1^2(1-c) & n_3 s + n_1 n_2(1-c) & -n_2 s + n_3 n_1(1-c) \\ -n_3 s + n_1 n_2(1-c) & c + n_2^2(1-c) & n_1 s + n_2 n_3(1-c) \\ n_2 s + n_3 n_1(1-c) & -n_1 s + n_2 n_3(1-c) & c + n_3^2(1-c) \end{bmatrix}, \tag{29}$$

where $s = \sin\theta$, $c = \cos\theta$. Denoting with R_{ij} the components of $\mathbf{R}$, the rotation of the tensor $\mathbb{H} = [H_{ijkl}]$ is performed according to the standard fourth order tensor rotation rule:

$$H_{ijkl} = H_{\alpha\beta\gamma\delta} R_{\alpha i} R_{\beta j} R_{\gamma h} R_{\delta l}. \tag{30}$$

4. Von Mises Distribution

Let us consider the case of the von Mises distribution for the fibers of a transversally isotropic material:

$$\rho\,(\Theta) = \frac{1}{2\pi I} \exp\left(b \cos 2\Theta\right), \tag{31}$$

where

$$I = \frac{1}{\pi} \int_0^{\pi} \exp\left(b \cos 2\Theta\right) d\Theta \tag{32}$$

and b is called *concentration parameter*.

The von Mises distribution is defined for $b > 0$, see Fig. 2(a). In particular, for b close to zero the von Mises distribution approaches an uniformly distributed density. In our model the distribution of the fibers is isotropic and the resulting material is isotropic. By increasing the concentration parameter b the von Mises distribution becomes unimodal and very close to a Gaussian distribution. For different values of the concentration parameter b the coefficients vary, i. e., $\kappa \in [0,1/3]$ and $\hat{k} \in [0,1/15]$, see Fig. 2(b). For $b > 4$ the parameter $\widehat{\kappa}$ becomes very small and the model described in (22) recovers the behavior of the distributed model proposed in (21) [4]. This means that the difference between the two models is relevant only for materials characterized by a certain degree of dispersion in the orientation and little fiber alignment. It is worth noting that when the distribution is uniformly distributed the resulting material is isotropic, therefore there is no need to adopt an anisotropic model. This is the case of tissues, for example the skin, where a greater stiffness in a particular direction is in general not required. Contrariwise, when the fibers are prevailingly aligned, strongly aligned models may be more indicated. This is the case of materials such as tendons. The proposed model works well for intermediate values of b. We are concerned mainly in biological tissues where no strong alignment, neither full isotropic orientation of the collagen fibers is observed. A typical example is the human cornea, where an average dispersion of the fibers has been documented through numerous experimental observations.

The expression of the average anisotropic contribution to the Piola-Kirchhoff stress tensor can be derived also using its definition (6) and (22):

$$\overline{\mathbf{S}}^{*}_{\text{aniso}} = <\overline{\mathbf{S}}_{\text{aniso}}> = \overline{\Psi}^{*}_{\text{aniso}} \left(\alpha \mathbf{H} + \beta \mathbb{H} : \overline{\mathbf{C}}\right) \tag{33}$$

where

$$\alpha = \sum_{j=0}^{3} a_j \overline{I}_4^{*j}, \quad \beta = \sum_{j=0}^{2} b_j \overline{I}_4^{*j}, \tag{34}$$

Explicit formulae for coefficients a_j, b_j are reported in Appendix.
The fourth order correlation tensor of the second Piola-Kirchhoff stress tensor, or covariance stress tensor, is defined as

$$\overline{\mathbf{R}}_{\text{aniso}} = <\overline{\mathbf{S}}_{\text{aniso}} \otimes \overline{\mathbf{S}}_{\text{aniso}}> - <\overline{\mathbf{S}}_{\text{aniso}}> \otimes <\overline{\mathbf{S}}_{\text{aniso}}> . \tag{35}$$

If we consider the linear approximation of the Helmholtz free energy (21) we obtain

$$\overline{\mathbf{S}}_{\text{aniso}} = 4M(\overline{I}_4)\,\overline{\Psi}_{\text{aniso}}(\overline{I}_4)\,\mathbf{A} \tag{36}$$

where we set $M(\overline{I}_4) = k_2(\overline{I}_4 - 1)$. In the limit of a linear approximation, we can write:

$$<\overline{\mathbf{S}}_{\text{aniso}}> \;=\; 4M(\overline{I}_4^*)\,\overline{\Psi}_{\text{aniso}}(\overline{I}_4^*)\,\mathbf{H}, \tag{37}$$

$$<\overline{\mathbf{S}}_{\text{aniso}} \otimes \overline{\mathbf{S}}_{\text{aniso}}> \;=\; 16\left[M(\overline{I}_4^*)\,\overline{\Psi}_{\text{aniso}}(\overline{I}_4^*)\right]^2 \mathbb{H}, \tag{38}$$

Therefore, the first approximation correlation tensor $\overline{\mathbf{R}}_{\text{aniso}}$ takes the form

$$\overline{\mathbf{R}}_{\text{aniso}} = \Delta(\mathbb{H} - \mathbf{H} \otimes \mathbf{H}), \qquad \Delta = 16\left[M(\overline{I}_4^*)\overline{\Psi}_{\text{aniso}}(\overline{I}_4^*)\right]^2 \tag{39}$$

or, in index notation

$$\left(\overline{\mathbf{R}}_{\text{aniso}}\right)_{ijkl} = \Delta\left(\mathbb{H}_{ijkl} - H_{ij}H_{kl}\right). \tag{40}$$

The non-zero elements of the correlation tensor $\overline{\mathbf{R}}_{\text{aniso}}$ are:

$$\begin{array}{rcl}
\overline{R}_{1111} = \overline{R}_{2222} & = & \Delta[3\widehat{\kappa} - \kappa^2] \\
\overline{R}_{3333} & = & \Delta[8\widehat{\kappa} - 4\kappa^2] \\
\overline{R}_{1212} = \overline{R}_{2121} = \overline{R}_{1221} = \overline{R}_{2112} & = & \Delta\,\widehat{\kappa} \\
\overline{R}_{1122} = \overline{R}_{2211} & = & \Delta[\widehat{\kappa} - \kappa^2] \\
\overline{R}_{2323} = \overline{R}_{3232} = \overline{R}_{2332} = \overline{R}_{3223} & = & \Delta[-4\widehat{\kappa} + \kappa] \\
\overline{R}_{2233} = \overline{R}_{3322} & = & \Delta[-4\widehat{\kappa} + 2\kappa^2] \\
\overline{R}_{3131} = \overline{R}_{1313} = \overline{R}_{3113} = \overline{R}_{1331} & = & \Delta[-4\widehat{\kappa} + \kappa] \\
\overline{R}_{3311} = \overline{R}_{1133} & = & \Delta[-4\widehat{\kappa} + 2\kappa^2].
\end{array} \tag{41}$$

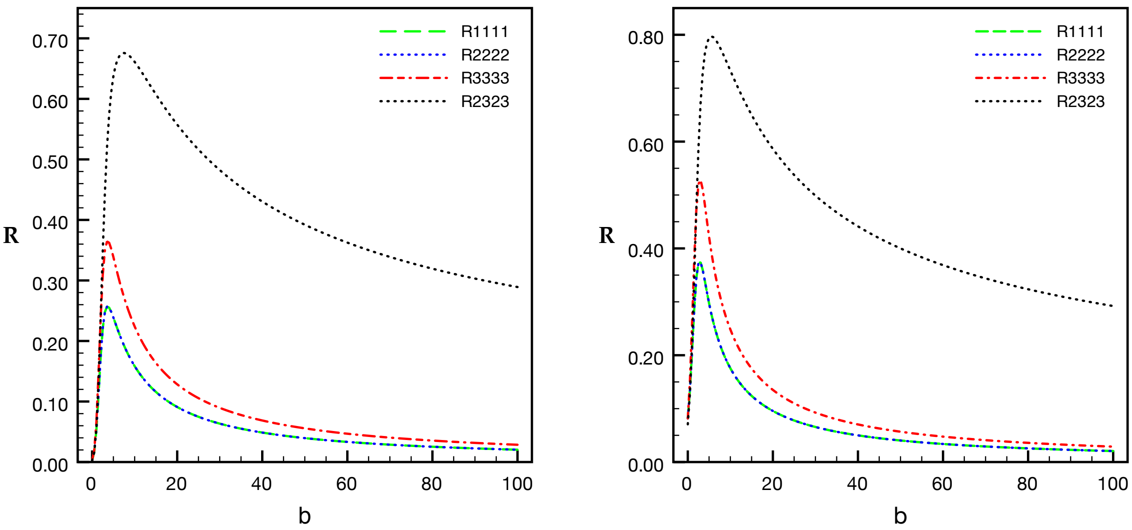

Fig. 3. Principal components of the correlation tensor (39) for varying concentration parameter b. (a) uniaxial loading; (b) equi-biaxial loading.

5. Numerical Tests

The model herein presented has been validated in uniaxial tension, simple shear and biaxial tension in [9]. In particular, comparison between the proposed approach and the model based on generalized structure tensors confirmed the validity of the second order approximation of the anisotropic strain energy density. In numerical tests we focused only on the contribution of the fibers, and did not consider the presence of an underlying matrix, neither a penalty contribution that accounts for volumetric changes. Therefore in (1) we assumed $\Psi_{\text{vol}} = 0$, $\Psi_{\text{iso}} = 0$, and accounted only for the isochoric response due to the fiber fraction. The model has been tested for soft materials such as the cornea, assuming for the stiffness parameters the values $k_1 = 1$, $k_2 = 1$, as well as stiffer biological tissues such as supra-spinatus tendon, adopting $k_1 = 5$, $k_2 = 30$. The evaluation of the mean values of the Piola-Kirchhoff stress, and of the corresponding Cauchy stress components, under uniaxial, biaxial and shear loading conditions and for different values of the concentration parameter b, are documented in [9]. Numerical tests have shown that, in all the loading cases for which GST models introduce large errors, the proposed approach guarantees a better performance, since it provides results closer to the ones furnished by a numerical integration over the distribution of the fiber orientation [9].

The components of the covariance stress tensor (39) depend on the concentration parameter b. The components of $\overline{\mathbf{R}}_{\text{aniso}}$ for varying b assume different values under uniaxial and equi-biaxial loading conditions. Fig. 3 shows the principal and shear components of the correlation tensor $\overline{\mathbf{R}}_{\text{aniso}}$ under uniaxial and equi-biaxial loading conditions for increasing values of the concentration parameter b. Plots evidence the nonlinear dependence of the standard deviation components on b with a prominent maximum. In particular, while for small values of b the standard deviation of the $\overline{\mathbf{S}}_{\text{aniso}}$ principal components is negligible, i. e., under isotropic condition the deviation from the mean value is small, the standard deviation increases with the concentration parameter, reaches a maximum, and then it decreases to zero for higher b. The shear component, instead, after reaching a maximum, maintains a large value also for high b.

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Appendix

Here we report explicit formulae for coefficients a_j, b_j in Eq. 34.

$$
\begin{aligned}
a_0 &= -4k_2 - 8\sigma_{\bar{I}_4}^2 k_2^3 - 12\sigma_{\bar{I}_4}^2 k_2^2 \\
a_1 &= 24\sigma_{\bar{I}_4}^2 k_2^3 + 12\sigma_I q^2 k_2^2 - 8k_2^2 \\
a_2 &= 16k_2^2 - 24\sigma_{\bar{I}_4}^2 k_2^3 \\
a_3 &= 8\sigma_{\bar{I}_4}^2 k_2^3 - 8k_2^2 \\
b_0 &= 4k_2 + 8k_2^2 \\
b_1 &= -16k_2^2 \\
b_2 &= 8k_2^2
\end{aligned}
$$